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Generalization of one inequality from IMO 2001

Arkady Alt [27]

ABSTRACT. Using only AM-GM inequality here was obtained all real positive k such, that inequality

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt[m]{1+kx_i}} \geq \frac{n+1}{\sqrt[m]{1+k}}, \tag{1}$$

holds for arbitrary $x_1, x_2, ..., x_{n+1} > 0$ subject to $x_1 x_2 ... x_{n+1} = 1$.
This inequality generalize inequality

$$\frac{1}{\sqrt{1+8x}} + \frac{1}{\sqrt{1+8y}} + \frac{1}{\sqrt{1+8z}} \geq 1, x, y, z > 0 \tag{2}$$

and $xyz = 1$, which, up to substitution $x = \frac{bc}{a}, y = \frac{ca}{b}, z = \frac{ab}{c}$, equivalent to well known inequality

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ca}} \geq 1, a, b, c > 0$$

(IMO 2001, problem 2).

Theorem. Let n and m be arbitrary natural numbers, $m \geq 2$ and let $k > 0$. Then inequality

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt[m]{1+kx_i}} \geq \frac{n+1}{\sqrt[m]{1+k}}, \tag{3}$$

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with equality condition $x_1 = x_2 = ... = x_{n+1} = 1$ holds for arbitrary real $x_1, x_2, ..., x_{n+1} > 0$,such that $x_1 x_2 ... x_{n+1} = 1$ if and only if $k \geq (n+1)^m - 1$.

Proof.
Necessity. Suppose that inequality (3) holds for arbitrary real $x_1, x_2, ..., x_{n+1} > 0$,such that $x_1 x_2 ... x_{n+1} = 1$,then in particularly it holds for $x_1 = x_2 = ... = x_n = x$ and $x_{n+1} = \frac{1}{x^n}$,where $x > 0$ arbitrary real number and we have

$$\frac{n}{\sqrt[m]{1+kx}} + \frac{n}{\sqrt[m]{1+\frac{k}{x^n}}} \geq \frac{n+1}{\sqrt[m]{1+k}} \implies$$

$$\implies \lim_{x\to\infty}\left(\frac{n}{\sqrt[m]{1+kx}} + \frac{1}{\sqrt[m]{1+\frac{k}{x^n}}}\right) \geq \frac{n+1}{\sqrt[m]{1+k}} \iff$$

$$1 \geq \frac{n+1}{\sqrt[m]{1+k}} \iff \sqrt[m]{1+k} \geq n+1 \iff k \geq (n+1)^m - 1.$$

To prove sufficiency we need to do some preparation, namely we need:

Lemma 1. For any natural $m \geq 2$ and any real $\theta > 1$ there is real positive β and p,such that
for any real positive t holds inequality

$$\sqrt[m]{1+(\theta^m - 1)\,t} \leq 1 + \beta t^p, \tag{4}$$

with equality condition $t = 1$.

Proof. Since

$$\sqrt[m]{1+(\theta^m - 1)\,t} \leq 1 + \beta t^p \iff 1 + (\theta^m - 1)\,t \leq (1+\beta t^p)^m \iff$$

$$(\theta^m - 1)\,t \leq \sum_{i=1}^{m} \binom{m}{i} \beta^i t^{ip}$$

we have to determine $\beta, p > 0$ such that latter inequality should be right for any $t > 0$, with equality condition $t = 1$.
Applying weighted AM-GM inequality to t^{ip}, $i = 1, 2, ..., m$ with the weights $\omega_i = \binom{m}{i}\beta^i$, $i = 1, 2, ..., m$,we obtain:

$$\sum_{i=1}^{m}(y\frac{m}{i})\beta^i t^{ip} = \sum_{i=0}^{m}\omega_i t^{ip} \geq W\left(\prod_{i=0}^{m} t^{i\omega_i p}\right)^{\frac{1}{W}} = Wt^{\frac{E}{W}},$$

where $W = \sum\limits_{i=1}^{m}\omega_i$ and $E = \sum\limits_{i=1}^{m} pi\omega_i$.

Since

$$\sum_{i=1}^{m}\omega_i = \sum_{i=1}^{m}\binom{m}{i}\beta^i = (1+\beta)^m - 1$$

and

$$\sum_{i=1}^{m} pi\omega_i = \sum_{i=1}^{m}\binom{m}{i}\beta^i ip =$$

$$= pm\beta\sum_{i=1}^{m}\binom{m-1}{i-1}\beta^{i-1} = pm\beta\sum_{i=0}^{m-1}\binom{m-1}{i}\beta^i = pm\beta\,(1+\beta)^{m-1},$$

we get $W = (1+\beta)^m - 1$ and $\dfrac{E}{W} = \dfrac{pm\beta\,(1+\beta)^{m-1}}{(1+\beta)^m - 1}$.

We claim $W = \theta^m - 1$ and

$$E = W \iff (1+\beta)^m - 1 = pm\beta\,(1+\beta)^{m-1} = \theta^m - 1,$$

and that imply $\beta = \theta - 1$ and $p = \dfrac{\theta^m - 1}{m\theta^{m-1}(\theta-1)}$;

Lemma 2. Let n be arbitrary natural number and let real $\beta \geq n$.Then for any positive real $a_1, a_2, ..., a_{n+1}$ holds inequality

$$\sum_{i=1}^{n+1}\frac{a_i^n}{a_i^n + \beta a_1 a_2 ... \hat{a}_i ... a_{n+1}} \geq \frac{n+1}{\beta+1}, \tag{5}$$

with equality condition $a_1 = a_2 = ... = a_{n+1}$.

$$\sum_{i=1}^{m}(y\frac{m}{i})\beta^i t^{ip} = \sum_{i=0}^{m}\omega_i t^{ip} \geq W\left(\prod_{i=0}^{m} t^{i\omega_i p}\right)^{\frac{1}{W}} = Wt^{\frac{E}{W}},$$

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with equality condition $a_1 = a_2 = ... = a_{n+1}$.

Proof. Applying AM-GM inequality we obtain:

$$\sum_{i=1}^{n+1} \frac{a_i^n}{a_i^n + \beta a_1 a_2 \dots \hat{a}_i \dots a_{n+1}} \geq \sum_{i=1}^{n+1} \frac{a_i^n}{a_i^n + \frac{\beta}{n} \cdot \sum\limits_{j=1, j\neq i}^{n+1} a_j^n} =$$

$$= \sum_{i=1}^{n+1} \frac{n a_i'}{\beta \sum\limits_{j=1}^{n+1} a_j' - (\beta - n) a_i'},$$

where $a_i' := a_i^{np}$, $i = 1, 2, ..., n+1$.
If $\beta = n$ we obtain

$$\sum_{i=1}^{n+1} \frac{n a_i'}{\beta \sum\limits_{j=1}^{n+1} a_j' - (\beta - n) a_i'} = 1;$$

Let $\beta > n$. Since,

$$\frac{n a_i'}{\beta \sum\limits_{j=1}^{n+1} a_j' - (\beta - n) a_i'}$$

is invariant of transformation

$$\left(a_1', a_2', ..., a_{n+1}'\right) \longmapsto \left(\tau a_1', \tau a_2', ..., \tau a_{n+1}'\right)$$

we can assume that $\sum\limits_{i=1}^{n+1} a_i' = 1$.
Let

$$S := \sum_{i=1}^{n+1} \frac{n a_i'}{\beta \sum\limits_{j=1}^{n+1} a_j' - (\beta - n) a_i'} = \sum_{i=1}^{n+1} \frac{n a_i'}{\beta - (\beta - n) a_i'}.$$

Then

$$S + \frac{n(n+1)}{\beta - n} = \sum_{i=1}^{n+1} \left(\frac{n a_i'}{\beta - (\beta - n) a_i'} + \frac{n}{\beta - n} \right) = \frac{\beta n}{\beta - n} \sum_{i=1}^{n+1} \frac{1}{\beta - (\beta - n) a_i'}$$

and, since

$$\sum_{i=1}^{n+1}\left(\beta-(\beta-n)\,a_i'\right)\cdot\sum_{i=1}^{n+1}\frac{1}{\beta-(\beta-n)\,a_i'}\geq(n+1)^2\,,$$

we obtain

$$S+\frac{n\,(n+1)}{\beta-n}\geq\frac{\beta n}{\beta-n}\cdot\frac{(n+1)^2}{\sum\limits_{i=1}^{n+1}\left(\beta-(\beta-n)\,a_i'\right)}=\frac{\beta\,(n+1)^2}{(\beta-n)\,(1+\beta)}\iff$$

$$S\geq\frac{\beta\,(n+1)^2}{(\beta-n)\,(1+\beta)}-\frac{n\,(n+1)}{\beta-n}=\frac{(n+1)\,(\beta n+\beta-n\beta-n)}{(\beta-n)\,(\beta+1)}=\frac{n+1}{\beta+1}.$$

Now we can prove sufficiency.
Let $k\geq(n+1)^m-1$ and $\theta:=\sqrt[m]{1+k}$, then $k=\theta^n-1$ and by lemma1 there is $\beta,\,p>0$ such that for any real $x>0$ holds inequality $\sqrt[m]{1+kx}=\sqrt[m]{1+(\theta^m-1)\,x}\leq1+\beta x^p$, where
$\beta=\theta-1=\sqrt[m]{1+k}\;-1\geq n.$

So, we have $\sum\limits_{i=1}^{n+1}\frac{1}{\sqrt[m]{1+kx_i}}\geq\sum\limits_{i=1}^{n+1}\frac{1}{1+\beta x_i^p}$ and since x_i can be represented as $x_i=\left(\frac{a_1a_2...\hat{a}_i...a_{n+1}}{a_i^n}\right)^{\frac{1}{p}},i=1,2,...n+1,$
then by Lemma 2 we obtain

$$\sum_{i=1}^{n+1}\frac{1}{\sqrt[m]{1+kx_i}}\geq\sum_{i=1}^{n+1}\frac{1}{1+\beta x_i^p}=\sum_{i=1}^{n+1}\sqrt[m]{\frac{a_i^n}{a_i^n+\beta a_1a_2...\hat{a}_i...a_{n+1}}}\geq$$

$$\geq\frac{n+1}{\beta+1}=\frac{n+1}{\sqrt[m]{1+k}}.$$

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As Corollary from Theorem we obtain by substitution that:

1. inequality

$$\sum_{i=1}^{n+1}\sqrt[m]{\frac{a_i^n}{a_i^n+ka_1a_2...\hat{a}_i...a_{n+1}}}\geq\frac{n+1}{\sqrt[m]{1+k}},\tag{6}$$

holds for any positive $a_1, a_2, ..., a_{n+1}$ iff $k \geq (n+1)^m - 1$. (substitution $x_i = \dfrac{a_1 a_2 ... \hat{a}_i ... a_{n+1}}{a_i^n}, i = 1, 2, ... n+1$ in inequality (3)); and

2. inequality

$$\sum_{i=1}^{n+1} \sqrt[m]{\frac{a_i}{a_i + k a_{i+1}}} \geq \frac{n+1}{\sqrt[m]{1+k}}, (a_{n+2} = a_1) \tag{7}$$

holds for any positive $a_1, a_2, ..., a_{n+1}$ iff $k \geq (n+1)^m - 1$. (substitution $x_i = \dfrac{a_{i+1}}{a_i}, i = 1, 2, ... n+1$ in inequality (3)).

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E-mail: arkady.alt@gmail.com